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Generalized Skew Derivations and Left Ideals in Prime and Semiprime Rings

Vannali Sreenivasulu[1] and Dr. R. Bhuvana Vijaya[2]
[1*]*Research Scholar, Department of Mathematics, Jawaharlal Nehru Technological University Anantapur, Ananthapuramu-515002, Andhra Pradesh, India. Email Id: sreelakshminivas@gmail.com*
[2]*Professor, Department of Mathematics, JNTUA College of Engineering, Anantapuramu-515002, Andhra Pradesh, India. Email Id:bhuvanarachamalla@gmail.com*

Abstract: *Let R be an associative ring, I a nonzero left ideal of R, G : R → R a generalized skew derivation with an associated nonzero skew derivation d and an automorphism α. In this paper, we study the following situations in prime and semiprime rings: (1) $G(x\ o\ y) = a\ (xy \pm yx)$; (2) $G[x,y] = a\ (xy \pm yx)$; (3) $d(x)od(y) = a(xy \pm yx)$; for all $x, y \in I$ and $a \in \{0,1,-1\}$.*

Key words: Prime rings, semiprime rings, skew derivations, generalized skew derivations, left ideals.

1. INTRODUCTION :

Throughout this paper, R is an associative ring, I a left ideal of R, $G : R \to$ R a generalized skew derivation associated with a nonzero skew derivation d and an automorphsm α of R. For any two elements $x,\ y \in R$, $[x,\ y]$ will denote the commutator element $xy - yx$ and $xoy = xy + yx$. We use extensively the following basic commutator identities: $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = [x, y]z + y[x, z]$. A ring R is said to be prime ring , if for any $x, y \in R, xRy = 0$ implies that either $x = 0$ or $y = 0$ and is called semiprime ring if for any $x \in R$, $xRx = 0$ implies $x = 0$. An additive mapping $d : R \to R$ is said to be a derivation of R if for any $x,\ y \in R$, $d(xy) = d(x)y + xd(y)$. By a skew derivation of R we mean an additive map d from R into itself which satisfies the rule $d(xy) = d(x)y + \alpha(x)d(y)$ for all $x, y \in R$ and α being an automorphism of R. For $\alpha = 1$ is the identity automorphism of R, d is known as a derivation of R. In particular, for a fixed $a \in R$, the mapping I_a : R → R given by $I_a(x) = [x, a]$ is a derivation called an inner derivation of R. An additive mapping $G : R \to R$ is called a generalized inner derivation if $G(x) = ax + xb$, for fixed $a, b \in R$. For such a mappings $G(xy) = G(x)y + x[y, b] = G(x)y + xI_b(y)$ for all $x, y \in R$. Motivated by the above observation, Bresar introduced the concept of generalized derivation as well as left multiplier mapping of R into R. The generalized derivation G of R is defined as an additive mapping $G : R \to R$ such that $G(xy) = G(x)y + xd(y)$ holds for any $x,\ y \in R$, where d is a derivation of R. So, every derivation is a generalized derivation, but the converse is not true in general. If $d = 0$, then we have $G(xy) = G(x)y$ for all $x,y \in R$, which is called a left multiplier mapping of R. Thus, generalized derivation generalizes both the concepts,. derivation on R. An additive mapping G : $R \to R$ is said to be a (right) generalized skew derivation of R if there exists a skew derivation d of R with an associated automorphism α such that $G(xy) = G(x)y + \alpha(x)d(y)$ holds for all $x, y \in R$.

In [1], Daif and Bell proved that if R is a semiprime ring with a nonzero ideal I and d is a derivation of R such that $d([x, y]) = \pm[x, y]$ for all $x,y \in I$, then I is central ideal. In particular, if $I = R$, then R is commutative. Recently, Quadri et al. [2] have generalized this result replacing derivation d with a generalized derivation in a prime ring R. In [3] Dhara has studied all the results of [2] in semiprime ring. Recently, Dhara and Patanayak [4] studied the results concerning generalized derivations. More precisely they studied the following cases in prime rings and semi prime rings; (1) $G(x\ o\ y) = a\ (xy \pm yx)$; (2) $G[x, y] = a\ (xy \pm yx)$; (3) $d(x)od(y) = a(xy \pm yx)$; for all $x, y \in I$ and $a \in \{0,1,-1\}$, where I being a left ideal of R and G is a generalized derivation of R. In the present paper, our aim is to discuss similar identities by taking G as a generalized skew derivation of R.

2. MAIN RESULTS :

Theorem 2.1. Let R be a semiprime ring and I a non-zero left ideal of R. If G is a genealized skew derivation of R associated with a non-zero skew derivation d and an automorphism α of R such that $G(x\ o\ y) = a\ (xy \pm yx)$ for all $x, y \in I$, where $a \in \{0, 1, -1\}$, then $\alpha[I, I]d(I) = 0$.

Proof : If $G(I) = 0$, then $0 = G(xy) = G(x)y + \alpha(x)d(y) = \alpha(x)d(y)$ for all $x, y \in I$ (2.1). Hence $\alpha(I)d(I) = 0$. Replacing x by $xz,\ z \in I$ in equation (2.1) we get $0 = \alpha(xz)d(y)$, for all

$x, y, z \in I$ (2.2). Again replacing x by zx, $z \in I$ in equation (2.1) we get $0 = \alpha(zx)d(y)$, for all $x, y, z \in I$ (2.3). Subtracting eq.(2.2) from (2.3) we get $0 = \{\alpha(zx) - \alpha(xz)\}d(y) = \alpha(zx - xz)d(y) = \alpha[x, z]d(y)$, for all $x, y, z \in I$. That is $0 = \alpha[I, I]d(I)$. Hence the result. So, let $G(I) \neq 0$, then as given $G(x\ o\ y) = a\ (xy \pm yx)$ for all $x, y \in I$ (2.4). Replacing y by yx, $x \in I$ in equation (2.4) we have $a\ (xyx \pm yx^2) = G(x\ o\ yx) = G\{(x\ o\ y)x - y[x, x]\} = G\{(x\ o\ y)x\} = G(x\ o\ y)x + \alpha(x\ o\ y)d(x) = a\ (xy \pm yx)x + \alpha(x\ o\ y)d(x) = a\ (xyx \pm yx^2) + \alpha(x\ o\ y)d(x)$. Finally, we get $\alpha(x\ o\ y)d(x) = 0$, $x, y \in I$ (2.5). Replacing y by zy, $z \in I$ in eq. (2.5) we get $0 = \alpha(x\ o\ zy)d(x) = \alpha\{z(x\ o\ y) + [x, z]y\}d(x) = \alpha(z)\alpha(x\ o\ y)d(x) + \alpha[x, z]\alpha(y)d(x) = 0 + \alpha[x, z]\alpha(y)d(x)$. Finally we get $\alpha[x, z]\alpha(y)d(x) = 0, x, y, z \in I$. That is $\alpha[I, I]\alpha(I)d(I) = 0$ (2.6). Since I is left ideal and α is an automorphism of R, we have $\alpha(I)$ is a left ideal of R. It follows $\alpha[I, I]R\alpha(I)d(I) = 0$ (2.7). Again R is a semiprime ring, then it must contain a family $\Omega = \{P_i : i \in \Lambda\}$ of prime ideals such that $\cap_{i\in\Lambda} P_i = \{0\}$. If P is a typical member of Ω and $x \in I$, we have either $\alpha[x, I] \subseteq P$ or $\alpha(I)d(x) \subseteq P$. For fixed P, the sets $T_1 = \{x \in I : \alpha[x, I] \subseteq P\}$ and $T_2 = \{x \in I : \alpha(I)d(x) \subseteq P\}$ form two additive subgroups of I such that $T_1 \cup T_2 = I$. Therefore, either $T_1 = I$ or $T_2 = I$. That is either $\alpha[I, I] \subseteq P$ or $\alpha(I)d(I) \subseteq P$. Both together gives us that $\alpha[I, I]d(I) \subseteq P$ for any $P \in \Omega$. Therefore $\alpha[I, I]d(I) \subseteq \cap_{i\in\Lambda} P_i = \{0\}$. That is $\alpha[I, I]d(I) = \{0\}$. □

Corollary 2.2. Let R be a prime ring and I a non-zero left ideal of R. If G is a genealized skew derivation of R associated with a non-zero skew derivation d and an automorphism α of R such that $G(x\ o\ y) = a\ (x\ o\ y)$ for all $x, y \in I$, where $a \in \{0, 1, -1\}$, then one of the following holds: (i) $\alpha(I)d(I) = 0$. (ii) R is commutative ring with char $(R) = 2$. (iii) R is commutative ring with char $(R) \neq 2$ and $G(x) = ax$ for all $x \in R$.

Proof: As in theorem-2.1, we have $\alpha[I, I]d(I) = \{0\}$. That is $\alpha[x, y]d(z) = 0$ for all $x, y, z \in I$ (2.8). Replacing y with wy, $w \in I$ in eq.2.8 we have $\alpha[x, wy]d(z) = 0$ $for\ all\ x, y, z, w \in I$.
$\alpha(w)\alpha[x, y]d(z) + \alpha[x, w]\alpha(y)d(z) = 0$ for all $x, y, z, w \in I$. Hence, $\alpha[x, w]\alpha(y)d(z) = 0$ for all $x, y, z, w \in I$. That is $\alpha[I, I]\alpha(I)d(I) = \{0\}$. Hence $\alpha[I, I]R\alpha(I)d(I) = \{0\}$ (2.9). Since R is prime ring, either $\alpha[I, I] = \{0\}$ or $\alpha(I)d(I) = \{0\}$. If $\alpha(I)d(I) = \{0\}$, we get conclusion (i). So, let $\alpha[I, I] = \{0\}$, that is $\alpha[x, y] = 0$ for all $x, y \in I$ (2.10). Replacing y with ry in eq.2.10 we have $\alpha[x, ry] = 0$ for all $x, y \in I$ and $r \in R$. That is $\alpha(r)\alpha[x, y] + \alpha[x, r]\alpha(y) = 0$ for all $x, y \in I$ and $r \in R$. Hence $\alpha[I, R]\alpha(I) = 0$. Again, this gives $0 = \alpha[RI, R] = \alpha[R, R]\alpha(I)$. Since left annihilator of a left-sided ideal is zero, we have $\alpha[R, R] = 0$, hence $[R, R] = 0$, that is R is commutative. If $\mathrm{char}(R) = 2$, we get conclusion (ii). So assume that $\mathrm{char}(R) \neq 2$. Then our assumption $G(x\ o\ y) = a\ (x\ o\ y)$ for all $x, y \in I$, where $a \in \{0, 1, -1\}$. That is $G(xy + yx) = a(xy + yx)$. Since R is commutative we have $2G(xy) = 2a(xy)$ for all $x, y \in I$. Since $\mathrm{char}(R) \neq 2$, then $G(xy) = a(xy)$ for all $x, y \in I$. Therefore $0 = G(xy) - a(xy) = G(x)y + \alpha(x)d(y) - axy = \{G(x) - ax\}y + \alpha(x)d(y)$ (2.11). Let $t \in I$, Since R is commutative, $xt \in I$. Replacing x by xt in eq.2.11, we have $0 = \{G(xt) - axt\}y + \alpha(xt)d(y) = \{G(x)t + \alpha(x)d(t) - axt\}y + \alpha(xt)d(y) = 0 + \alpha(xt)d(y)$.That is $\alpha(xt)d(y) = 0$ for all $x, t, y \in I$ (2.12). Replacing y by ys, $s \in R$ in eq.2.12, we get $0 = \alpha(xt)d(ys) = \alpha(xt)d(y)s + \alpha(xt)\alpha(y)d(s) = 0 + \alpha(xt)\alpha(y)d(s) = \alpha(xt)\alpha(y)d(s)$. Replacing y by yr, $r \in R$, we get $0 = \alpha(xty)\alpha(r)d(s)$, that is $\alpha(I^3)Rd(R) = (0)$. Since R is prime we have either $\alpha(I^3) = 0$ or $d = 0$. If $\alpha(I^3) = 0$ implies $I^3 = 0$, since R is prime this forces $I = 0$. Which is a contradiction. Hence $d = 0$.
From eq.2.11 we have $\{G(x) - ax\}y = 0$ for all $x, y \in I$, that is $\{G(x) - ax\}I = 0$ for all $x \in I$ which yields $G(x) = ax$ for all $x \in I$. Replace x by rx, $r \in R$, we have $G(rx) = arx$, that is $\{G(r) - ar\}x = 0$, for all $x \in I, r \in R$. Hence $G(r) = ar$, for all $r \in R$.
□

Theorem 2.3. Let R be a semiprime ring and I a non-zero left ideal of R. If G is a genealized skew derivation of R associated with a non-zero skew derivation d and an automorphism α of R such that $G[x, y] = a\ (xy \pm yx)$ for all $x, y \in I$, where $a \in \{0, 1, -1\}$, then $\alpha[I, I]d(I) = 0$.

Proof: If $G(I) = 0$, then $0 = \mathrm{G}(\mathrm{I}^2) = \mathrm{G}(\mathrm{I})\mathrm{I} + \alpha(\mathrm{I})\mathrm{d}(\mathrm{I}) = \alpha(\mathrm{I})\mathrm{d}(\mathrm{I})$ and hence as in theorem 2.1 we have $\alpha[I, I]d(I) = 0$, which is our conclusion. So assume that $G(I) \neq 0$. Then by our assumption we have $G[x, y] = a\ (xy \pm yx)$ for all $x, y \in I$, where $a \in \{0, 1, -1\}$ (2.13). Put $y = yx$ in equ. (2.13) and we get $G[x, yx] = a\ (xyx \pm yxx)$, that is $G\{y[x, x] + [x, y]x\} = a\ (xy \pm yx)x$, this implies $G\{[x, y]x\} = a\ (xy \pm yx)x$, implies $G[x, y]x + \alpha[x, y]d(x) = a\ (xy \pm yx)x$. Now using equ.(2.13) we have $a\ (xy \pm yx)x + \alpha[x, y]d(x) = a\ (xy \pm yx)x$, this yields $\alpha[x, y]d(x) = 0$ for all $x, y \in I$ (2.14). Again putting $y = zy$ where $z \in I$ in eq. 2.14, we have $\alpha[x, zy]d(x) = 0$, implies

$\alpha(z)\alpha[x,y]d(x) + \alpha[x,z]\alpha(y)d(x) = 0$. Hence we get using eq.2.14, $\alpha[x,z]\alpha(y)d(x) = 0$ for all $x,y,z \in I$ (2.15). That is $\alpha[I,I]\alpha(I)d(I) = 0$ which is same as eq. 2.6 in theorem-2.1. By same argument as in theorem-2.1, we conclude the result. □

Corollary 2.4. Let R be a prime ring and I a non-zero left ideal of R. If G is a genealized skew derivation of R associated with a non-zero skew derivation d and an automorphism α of R such that $G[x,y] = a\,(xy \pm yx)$ for all $x,y \in I$, where $a \in \{0,1,-1\}$, then one of the following holds: (i) $I[I,I] = 0$. (ii) $G(x) = \mp ax$ for all $x \in I$. In case $G(x) = -ax$ for all $x \in I$, with $a \neq 0$, then $char(R) = 2$.

Proof : By theorem - 2.3 we may conclude that $\alpha[I,I]d(I) = 0$. Then by same argument as given in corollary - 2.2, we get either R is commutative or $\alpha(I)d(I) = 0$. Let R is non-commutative, then for any $x,y \in I$, we have $G(xy) = G(x)y + \alpha(x)d(y) = G(x)y$, that is G acts as a left multiplier on I. Then for any $x,y,z \in I$, replacing y with yz in our hypothesis $G[x,y] = a\,(xy \pm yx)$ for all $x,y \in I$, where $a \in \{0,1,-1\}$, we have $G[x,yz] = a\,(xyz \pm yzx)$, that is $G\{y[x,z] + [x,y]z\} = a\{(xy \pm yx)z \mp y[x,z]\}$ for all $x,y,z \in I$ (2.16). Since G acts as a left multiplier map on I, this implies $G\{y[x,z] + [x,y]z\} = a\{(xy \pm yx)z \mp y[x,z]\}$. That is $G(y)[x,z] + G[x,y]z = a(xy \pm yx)z \mp ay[x,z]$. That is $G(y)[x,z] + a\,(xy \pm yx)z = a(xy \pm yx)z \mp ay[x,z]$. Hence $G(y)[x,z] = \mp ay[x,z]$. That is $\{\mathrm{G(y)} \pm \mathrm{ay}\}[x,z] = 0$, for all $x,y \in I$. Replacing y with yu, where $u \in I$, we find that $\{\mathrm{G(y)} \pm \mathrm{ay}\}u[x,z] = 0$, for all $x,y,u,z \in I$. Again replacing u with ru, where $r \in R$, we have $\{\mathrm{G(y)} \pm \mathrm{ay}\}RI[x,z] = 0$, for all $x,y,z \in I$. Since R is prime, either $I[I,I] = 0$ or $G(y) = \mp ay$ for all $y \in I$. When $G(y) = -ay$, our assumption $G[x,y] = a\,(xy + yx)$ implies $-a[x,y] = a(xy + yx)$ $x,y \in I$. This implies $2axy = 0$. Replacing x with rx, $r \in R$, we have $2arxy = 0, x,y \in I$ and $r \in R$, that is $2aRI^2 = 0$ implies $char(R) = 2$.

□

Theorem 2.5. Let R be a semi-prime ring and I a non-zero left ideal of R. If d is a non-zero skew derivation of R such that $d(x)od(y) = a(xy \pm yx)$ for all $x,y \in I$, where $a \in \{0,1,-1\}$, then $\alpha(I)[x,d(x)]_2 = 0$. In case $I = R$ and R is 2-torsion free, d maps R into its center.

Proof : We have for all $x,y \in I, d(x)d(y) + d(y)d(x) = a(xy \pm yx)$ (2.17). Replacing y with yx, $x \in I$, we have $d(x)d(yx) + d(yx)d(x) = a(xyx \pm yxx)$. That is $d(x)\{d(y)x + \alpha(y)d(x)\} + \{d(y)x + \alpha(y)d(x)\}d(x) = a(xy \pm yx)x$. This implies

$$d(x)d(y)x + d(x)\alpha(y)d(x) + d(y)xd(x)$$
$$+\alpha(y)d(x)^2 = d(x)d(y)x + d(y)d(x)x.$$

That is $d(x)\alpha(y)d(x) + d(y)[x,d(x)] + \alpha(y)d(x)^2 = 0$ (2.18). Put $y = xy$ in equ.(2.18), we have $d(x)\alpha(y)d(x) + d(y)[x,d(x)] + \alpha(y)d(x)^2 = 0$. That is $d(x)\alpha(x)\alpha(y)d(x) + \{d(x)y + \alpha(x)d(y)\}[x,d(x)] + \alpha(x)\alpha(y)d(x)^2 = 0$ (2.19). Left multiplying equ.(2.18) by $\alpha(x)$ and then subtracting from equ.(2.19), we get

$$\begin{aligned} d(x)\alpha(x)\alpha(y)d(x) & \\ &+ \{d(x)y + \alpha(x)d(y)\}[x,d(x)] \\ &+ \alpha(x)\alpha(y)d(x)^2 \\ &- \alpha(x)d(x)\alpha(y)d(x) \\ &- \alpha(x)d(y)[x,d(x)] \\ &- \alpha(x)\,\alpha(y)d(x)^2 = 0 \end{aligned}$$

That is, $[d(x),\alpha(x)]\alpha(y)d(x) + d(x)y[x,d(x)] = 0$. Replacing $\alpha(y)$ by y and $\alpha(x)$ by x in above equation, $[d(x),x]yd(x) + d(x)y[x,d(x)] = 0$ (2.20). Replacing y by $d(x)y$ in eq.2.20, $[d(x),x]d(x)yd(x) + d(x)d(x)y[x,d(x)] = 0$ (2.21). Left multiplying eq.2.20 with $d(x)$ and then subtracting from eq.2.21, we have $[[d(x),x],d(x)]yd(x) = 0$ (2.22). Again replacing y by $y[d(x),x]$ in eq.2.22, we have $[[d(x),x],d(x)]y[d(x),x]\,d(x) = 0$ (2.23). Right multiplying eq. (2.22) by $[d(x),x]$ and then subtracting from eq. 2.23, we get $[[d(x),x],d(x)]y[d(x),x]d(x) - [[d(x),x],d(x)]yd(x)[d(x),x] = 0$, implies $[[d(x),x],d(x)]y\big[[d(x),x],d(x)\big] = 0$. Replacing y by $\alpha(ry)$, $r \in R$ in this eq. we get $[[d(x),x],d(x)]\alpha(ry)\big[[d(x),x],d(x)\big] = 0$, that is $[[d(x),x],d(x)]\alpha(r)\alpha(y)\big[[d(x),x],d(x)\big] = 0$. Hence we get $[[d(x),x],d(x)]R\alpha(y)\big[[d(x),x],d(x)\big] = 0$. Left multiplying this eq. with (y), $\alpha(y)[[d(x),x],d(x)]R\alpha(y)\big[[d(x),x],d(x)\big] = 0$. Now R is semiprime ring, this forces $\alpha(y)[[d(x),x],d(x)] = 0$, for every $x,y \in I$. That is $\alpha(I)[[d(x),x],d(x)] = 0$ or $\alpha(I)[[x,d(x)],d(x)] = 0$. That is $\alpha(I)[x,d(x)]_2 = 0$. In case $I = R$, $[x,d(x)]_2 = 0$ for all $x \in R$, and then by [5], $d(R) \subseteq Z(R)$. □

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